ELSEVIER Stochastic Processes and their Applications 71 (1997) 275–284

stochastic processes and their applications

Exact probabilities and asymptotics for the one-dimensional coalescing ideal gas[1]

A. Ermakov*

CWI, Kruislaan 413, 1098 SJ Amsterdam, Netherlands

Received 20 March 1997; received in revised form 27 June 1997

Abstract

We consider a modification of the well-known system of coalescing random walks in one dimension, both in discrete and continuous time. In our models each particle moves with unit speed, and it can change its direction of movement only at times of collisions with other particles. At these times (and at time 0) the direction is chosen randomly, with equal probability to the left or to the right, independently of anything else. In this article, we exhibit the exact distributions of particle density and of other relevant quantities at finite time t, and their asymptotics as $t \to \infty$. In particular, it appears that the density of particles at time t is equal to the probability of the event that a simple random walk starting at site one first hits the origin after time t. It is noteworthy that a relation of the same kind is known to hold for the standard system of coalescing random walks in one dimension, though the proof is quite different in that case. © 1997 Elsevier Science B.V.

Keywords: Interacting particle systems; Coalescing random walks; Clustering; Asymptotic density

1. Introduction and statement of results

Before introducing the coalescing ideal gas model, let us first consider the system of *coalescing random walks* (CRW), which is one of the simplest interacting particle systems. Its state space is $\mathbb{Z}^* = \{\text{all subsets of } \mathbb{Z}^1\}$, and time is continuous. At time 0 there is a particle in every site. Each particle performs an independent, continuous-time simple random walk with jump rate 1, until it runs into another particle. When two particles meet, they coalesce into one particle, which resumes the same random walk. The behaviour of this system is well understood (see, for instance, Harris, 1976; Griffeath, 1979; Bramson and Griffeath, 1980b; Arratia, 1981). Let S_t be the position of a continuous time simple random walk at time t: it starts at 0 and makes jumps to each of the two neighbour sites at rate $\frac{1}{2}$ (so the total jump rate is 1). Let $\tau(x)$, $x \in \mathbb{Z}$ be the first hitting time of site x:

$$\tau(x) = \inf\{t:\ S_t = x\}.$$

* E-mail: a.ermakov@cwi.nl.

[1] Supported by the Netherlands Organization for Scientific Research (NWO).

PII S0304-4149(97)00077-X

Let $\rho_t(\mathrm{CRW})$ be the "particle density", or the probability that the origin in the system of coalescing random walks described above is occupied at time t. The value of $\rho_t(\mathrm{CRW})$ can be computed using a duality relation (Harris, 1976; Bramson and Griffeath, 1980a): it is equal to the probability that in the classical one-dimensional *voter model* (see, e.g., Liggett, 1985), the opinion of the individual at the origin survives at time t. Let n_t be the size at time t of the interval of $\mathbb{Z}$ made up of those sites which have inherited the opinion of $(0,0)$. n_t can be viewed as a rate-2 simple random walk on $\mathbb{Z}^+$ (or birth and death process) starting at 1, with absorption at 0.

Hence,

$$\begin{aligned}\rho_t(\mathrm{CRW}) &= P(n_t>0) = P(\tau(-1)>2t)\\ &= \mathrm{e}^{-2t}(I_0(2t)+I_1(2t)) \sim \frac{1}{\sqrt{\pi}}t^{-1/2}, \quad t\to\infty. \end{aligned} \tag{1}$$

The third equality can be obtained by using the Laplace transform (or more directly, see Feller, 1968b, p. 60). $I_\nu(t)$ is the modified Bessel function of the first kind:

$$I_\nu(z)=\sum_{m=0}^{\infty}\frac{(z/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)}. \tag{2}$$

The asymptotics for the particle density of CRW in higher dimensions was obtained in Bramson and Griffeath (1980a); duality relations are also exploited, but in this case the relation between CRW and simple random walk (the second equality in (1)) is distorted by a random factor and is hardly usable at all.

In Bramson and Griffeath (1980b) it was noted that the asymptotics of $\rho_t(\mathrm{CRW})$ (1) remains the same if we start the CRW from an initial configuration distributed according to an arbitrary mixing measure $\mu\neq\delta_\emptyset$ on $\mathbb{Z}^*$ where $\delta_\emptyset$ denotes the measure which assigns measure 1 to the empty set.

It is thus natural to ask, to which extent the asymptotics of $\rho_t(\mathrm{CRW})$ can be influenced by introducing local dependence in the dynamics of the underlying coalescing random walk. One simple example of such process is the following: We start at time 0 from $2\mathbb{Z}$, i.e., from the configuration in which all even sites are occupied by particles, and assign to each particle a direction of movement: either $+1$ or -1, with equal probabilities, independently. The particles begin moving in the assigned directions with unit speed, and each particle keeps its direction until it collides with another particle. At the collision, the two particles coalesce into one particle, which chooses its direction with equal probability in each way, independent of anything else, and moves further in this direction. We shall call this interacting particle system *coalescing ideal gas* (CIG), and denote the set of x-coordinates of the particles at time $i\in\mathbb{Z}^+$ by $\hat{\xi}_i$. (Since all the collisions in this process occur at integer times, we shall restrict to integer times.) We shall also consider the continuous space and time coalescing ideal gas ξ_t^λ, $\lambda\in(0,\infty)$, $t\in[0,\infty)$ which has the same dynamics as described above, but its initial distribution is a Poisson point process of density λ on $\mathbb{R}^1$. Here and below we mark discrete processes and variables with a hat, to distinguish them from the continuous ones.

This system can be considered as a highly simplified model for aggregation processes at low pressure, such as an early stage of polymerisation in gas phase or aggregation of mist particles. The chemical bonds in the first and the surface tension in the second case make the particles coalesce when they meet, and the absence of air lets them move forward without deviating.

If we change the collision rule from coalescence to annihilation, we obtain the *annihilating ideal gas*, or the *deterministic surface growth* model. It has random initial state and deterministic dynamics. This model and its scaling limit are rather good understood (see Belitsky and Ferrari, 1995 and references therein). Fisch (1992) has found the asymptotic rate of the particle density, and conjectured that for the coalescing case it is the same. This conjecture is now proved in Theorem 1 in this paper.

First, we examine the discrete system $\hat{\xi}_t$. All the particles have even coordinates at even times, and odd coordinates at odd times.
Introduce

$$\hat{\theta}_i = P \text{ (the particle which started from the origin at time 0 has its first collision with another particle at time } i), \ i > 0,$$

$$\hat{\varphi}_i = P \text{ (a collision takes place at time } i \text{ at } x = (i \bmod 2)), \ i > 0,$$

$$\hat{\rho}_i = P \text{ (there is a particle at time } i \text{ at } x = (i \bmod 2)) = P\ (\hat{\xi}_i \cap \{0,1\} \neq \emptyset), \ i \geqslant 0.$$

Here $(i \bmod 2)$ denotes the remainder of dividing i by 2. For convenience, we define $\hat{\theta}_0 = 0$, $\hat{\varphi}_0 = 1$.

We shall also need an (independent of $\hat{\xi}_t$) discrete time *simple random walk* $\hat{S}_n$ on $\mathbb{Z}$ starting at 0.

$$X_i = \text{i.i.d.}, \quad P(X_i = -1) = P(X_i = 1) = \tfrac{1}{2}, \quad i \in \mathbb{Z}^+,$$

$$\hat{S}_0 = 0, \qquad \hat{S}_n = \sum_{i=0}^{n-1} X_i.$$

Let $\hat{\tau}_k(x)$ be the *k-th return time* of $\hat{S}_n$ to the site $x \in \mathbb{Z}$:

$$\hat{\tau}_1(x) = \min\{i > 0:\ \hat{S}_i = x\},$$

$$\hat{\tau}_k(x) = \min\{i > \hat{\tau}_{k-1}(x):\ \hat{S}_i = x\}, \quad k > 1.$$

In the discrete CIG, the increments of a trajectory of a particle are distributed like X_i, but they are positively correlated: the covariance of two increments is equal to the probability that there is no collision inbetween. One could expect therefore that the particles in CIG will aggregate at a higher rate, and the density of the particles will be asymptotically smaller than that of CRW, but this is not the case. In Section 2 we shall prove

Theorem 1. *For the discrete coalescing ideal gas the following relations hold:*

$$\text{for } n \geqslant 1, \quad \hat{\theta}_n = P(\hat{\tau}_1(0) = 2n) = \frac{1}{2n-1}\binom{2n}{n} 2^{-2n} \underset{n\to\infty}{\sim} \frac{1}{2\sqrt{\pi}} n^{-3/2}, \tag{3}$$

$$\hat{\varphi}_n = 2\hat{\theta}_{n+1} = \frac{1}{n+1}\binom{2n}{n} 2^{-2n} \underset{n\to\infty}{\sim} \frac{1}{\sqrt{\pi}} n^{-3/2}, \tag{4}$$

$$\begin{aligned} \hat{\rho}_n &= 2P(\hat{\tau}_1(0) > 2(n+1)) = 2P(\hat{\tau}_1(-1) > 2(n+1)) \\ &= \binom{2n+1}{n+1} 2^{-2n} \underset{n\to\infty}{\sim} \frac{2}{\sqrt{\pi}} n^{-1/2}. \end{aligned} \tag{5}$$

Note the similarity between (5) and (1), which we shall discuss later.

The key step in the proof of the Theorem 1 will be establishing a coupling relation which yields the first equality in (3).

We now proceed with the continuous version of CIG, ξ_t^λ, which was introduced above. Let $\mathring{\xi}_t^\lambda$ be the Palm version of this process, which is obtained by adding a particle at the origin to the initial configuration ($\mathring{\xi}_0^\lambda = \xi_0^\lambda \cup \{0\}$) and then using the same dynamics. Let P_λ and $\mathring{P}_\lambda$ be the probability measures associated with these processes.

We define the following densities which characterise these processes:

$$\theta_t^\lambda \, \mathrm{d}t = \mathring{P}_\lambda \text{ (the particle which started from the origin at time 0 has its first collision with another particle at time } s \in [t, t+\mathrm{d}t]),$$

$$\varphi_t^\lambda \, \mathrm{d}x \, \mathrm{d}t = P_\lambda \text{ (there is a collision in the space–time area } [0, \mathrm{d}x] \times [t, t+\mathrm{d}t]),$$

$$\begin{aligned} \rho_t^\lambda \, \mathrm{d}x &= P_\lambda \text{ (there is a particle at time } t \text{ in the interval } [0, \mathrm{d}x]) \\ &= P(\xi_t^\lambda \cap [0, \mathrm{d}x] \neq \emptyset). \end{aligned}$$

It is clear that θ_t^λ, φ_t^λ and ρ_t^λ are finite, positive, continuous functions, and that the process ξ_t^λ is space-stationary, so that these definitions are consistent.

Now we formulate the continuous analog of Theorem 1:

Theorem 2. *For the processes* $\mathring{\xi}_t^\lambda$, ξ_t^λ *(continuous* CIG) *the following relations hold:*

$$\theta_t^\lambda = \lambda \mathrm{e}^{-\lambda t}[I_0(-\lambda t) + I_1(-\lambda t)] \underset{t\to\infty}{\sim} \frac{1}{2\sqrt{2\pi\lambda}} t^{-3/2}, \tag{6}$$

$$\varphi_t^\lambda = \lambda t^{-1} \mathrm{e}^{-\lambda t} I_1(\lambda t) \underset{t\to\infty}{\sim} \sqrt{\frac{\lambda}{2\pi}} t^{-3/2}, \tag{7}$$

$$\rho_t^\lambda = \lambda P(\tau(-1) \geqslant \lambda t) = \lambda \mathrm{e}^{-\lambda t}[I_0(\lambda t) + I_1(\lambda t)] \underset{t\to\infty}{\sim} \sqrt{\frac{2\lambda}{\pi}} t^{-1/2}, \tag{8}$$

where $I_\nu(z)$ *is defined by* (2).

Note that although at $t \to 0$, both θ_t^λ and ρ_t^λ are clearly linear in λ:

$$\lim_{t \downarrow 0} \theta_t^\lambda = \rho_0^\lambda = \lambda,$$

this linearity is not preserved when t is big.

It is not surprising that both continuous and discrete CIG have essentially the same asymptotics. More remarkable is the fact that the asymptotics of the particle densities of these two versions of CIG, $\hat{\rho}_n$ and ρ_t^λ, are, up to a constant, equal to those of the particle density $\rho_t(\text{CRW})$ (1), despite the higher mean square displacements of the particles in CIG. Furthermore, all the three densities ($\hat{\rho}_n$, ρ_t^λ and $\rho_t(\text{CRW})$) can be expressed through return times of a simple random walk (see (1), (5) and (8)). However, while (1) follows from a natural coupling between the processes, and another less direct coupling exists in (3), we do not see a direct probabilistic relation between CIG and the simple random walk, which yields the first equality in (8).

Along with the asymptotic similarity of the densities of CIG and CRW, there is also a fundamental difference between them: unlike the latter system (Bramson and Griffeath, 1980b, Theorem 1), the asymptotics of our system are sensitive to the initial condition, i.e., to the initial density of particles.

2. Further properties and proofs for the discrete case

Lemma 3. *Consider the trajectory of a* CIG *particle which starts at site* x_0 *at time* $t_0 = 0$. *Let* (x_n, t_n), $n \geqslant 1$ *be the coordinates of the n-th collision of the particle. Then*

1. $(x_{n+1} - x_n)_{n \in \mathbb{Z}^+}$ *(and hence also* $(t_{n+1} - t_n)_{n \in \mathbb{Z}^+}$*) is an i.i.d. sequence,*
2. $P(x_{n+1} - x_n = k) = \frac{1}{2} P(t_{n+1} - t_n = |k|) = \frac{1}{2} \hat{\theta}_{|k|}$, (9)
3. $\hat{\varphi}_n = 2\hat{\theta}_{n+1}, \quad n \geqslant 0,$ (10)
4. $\hat{\rho}_n = \sum_{i > n} \hat{\varphi}_i, \quad n \geqslant 0.$ (11)

Proof. The system of CIG is based on an underlying collection of i.i.d. random variables

$$\{\alpha_{x,t}\colon (x,t) \in \mathbb{Z} \times \mathbb{Z}^+, x + t \text{ is even}\}, \quad \text{such that} \quad P(\alpha_{x,t} = 1) = P(\alpha_{x,t} = -1) = 1/2. \tag{12}$$

If there is a collision at (x,t) then after it the particle takes the direction $\alpha_{x,t}$.

Fix an integer $n > 0$ and a sequence $\{a_i\}_{i=1}^{n+1}$. Let $b_i = |a_i|$, $a = \sum_{i=1}^n a_i$, $b = \sum_{i=1}^n b_i$. By translation invariance we can assume that $x_0 = -(a+b)$. We have to show that the events

$$A = \{x_i - x_{i-1} = a_i,\ i = 1, \ldots, n\}$$

and

$$B = \{x_{n+1} - x_n = a_{n+1}\}$$

are independent of each other. Note that A implies $(x_n, t_n) = (x_0 + a, b) = (-b, b)$. Assume, without loss of generality, that $\alpha_{-b,b} = +1$. Let

$$t^* = \min\{t \in \mathbb{Z}^+ \colon \text{ there is a collision at } (t+2, t) \text{ and } \alpha_{t+2,t} = -1\}.$$

Under our assumptions, if $(x_n, t_n) = (-b, b)$ then we have

$$x_{n+1} - x_n = t^* + 1. \tag{13}$$

The event A is independent of $\{\alpha_{x,t} \colon x > 0\}$, since any particle which passes through the region $\{(x,t) \colon x > 0\}$ cannot visit the point $(-b, b)$. On the other hand, t^* is independent of $\{\alpha_{x,t} \colon x \leqslant 0\}$. Hence, t^* is independent of A, and so is B, because of (13). This proves assertion 1 of the lemma.

Now, we drop all the assumptions made above and instead assume that $\alpha_{0,0} = +1$. The definition of t^* remains the same. One can see that the first collision of the particle which starts at the origin occurs at time $t^* + 1$. Hence,

$$\hat{\theta}_{i+1} = P(t^* = i).$$

This observation together with (13) proves the last equation in (9).

From the definition of t^* it is clear that

$$\{t^* = i\} = \{\text{there is a collision at } (i+2, i)\} \cap \{\alpha_{i+2,i} = -1\}.$$

Taking the probabilities we obtain $\hat{\theta}_{i+1} = 1/2\hat{\varphi}_i$, which proves (10).

Eq. (11) follows from the individual ergodic theorem and the fact that each collision replaces two particles by one, so that the particle density decreases each time by $\hat{\varphi}_n$. □

Proof of Theorem 1. We shall prove (3) by constructing a bijection between the two configuration spaces.

Let T be the time of the first collision of the CIG particle which starts from the origin. Assume for convenience that $\alpha_{0,0} = +1$. Then all the $T+1$ particles which start from $0, 2, \ldots, 2T$ coalesce by time T into a single particle with the coordinates (T, T). These particles undergo exactly T collisions up to time T, since each collision reduces the number of particles by one. Let

$$R_T = \{(2i, 0),\ i = 0, \ldots, T\} \cup \{\text{the points of collisions, up to time } T, \text{ of the particles which visit } (T,T)\}.$$

We have just seen that $\operatorname{card}(R_T) = 2T + 1$.

Let us now order the elements of R_T:

$$R_T = \{(\tilde{x}_0, \tilde{t}_0), (\tilde{x}_1, \tilde{t}_1), \ldots, (\tilde{x}_{2T}, \tilde{t}_{2T})\} \tag{14}$$

in such way that

$$\{i < j\} \text{ iff } \{\tilde{x}_i + \tilde{t}_i < \tilde{x}_j + \tilde{t}_j \text{ or } (\tilde{x}_i + \tilde{t}_i = \tilde{x}_j + \tilde{t}_j \text{ and } \tilde{t}_i < \tilde{t}_j)\}.$$

Such an ordering exists and is unique.

Consider the simple random walk

$$\tilde{S}_n = \sum_{i=0}^{n-1} \alpha_{\tilde{x}_i,\tilde{t}_i}, \quad \tilde{S}_0 = 0.$$

(We have assumed that $\alpha_{0,0} = 1$, and hence $\tilde{S}_1 = 1$. The other case can be treated similarly.)

The mapping between R_T and $(\tilde{S}_i)_{i=0}^{2T}$ is bijective, since, from knowing $(\tilde{x}_j, \tilde{t}_j)_{j=0}^{i}$ and $(\alpha_{\tilde{x}_j,\tilde{t}_j})_{j=0}^{i}$, $i<2T$ one can determine $(\tilde{x}_{i+1}, \tilde{t}_{i+1})$. Note that $2T$ is the first return time of the random walk $\tilde{S}_i$ to the origin. Indeed, each step up $(\alpha_{\tilde{x}_i,\tilde{t}_i} = +1)$ means that the next particle in the ordering (14), $(\tilde{x}_{i+1}, \tilde{t}_{i+1})$ is a new particle with $\tilde{t}_{i+1} = 0$, so we can say that such a step adds one particle to the system. On the other side, a step down $(\alpha_{\tilde{x}_i,\tilde{t}_i} = -1)$ means that a collision takes place at $(\tilde{x}_{i+1}, \tilde{t}_{i+1})$: $\tilde{t}_{i+1} > 0$ and the number of particles decreases by one. Hence, the number of CIG particles after i steps of construction is equal to $\tilde{S}_i + 1$. At step $2T$ the number of particles reduces to one, and the random walk returns to zero for the first time. Therefore,

$$\hat{\theta}_n = P(T = n) = P(\hat{\tau}_1(0) = 2n),$$

which yields (3).

This, with (10), gives (4). The first equation in (5) follows from (3), (10) and (11). The second equation follows from the reflection principle (Feller, 1968a, p. 77). □

3. Proofs for the continuous case

In order to prove Theorem 2 we shall establish the continuous analogs of some relations in Lemma 3 and add one more relation (15), which is necessary to resolve the system.

Lemma 4. *For the processes* $\overset{\circ}{\xi}{}_t^\lambda$, ξ_t^λ *the following relations hold*:

$$\theta_t^\lambda = \lambda\, \mathrm{e}^{-2\lambda t} + \int_0^t \varphi_s^\lambda \mathrm{e}^{-2\lambda(t-s)}\, \mathrm{d}s, \tag{15}$$

$$\varphi_t^\lambda = \frac{\lambda}{2}\theta_t^\lambda + \frac{1}{2}\int_0^t \varphi_s^\lambda \theta_{t-s}^\lambda\, \mathrm{d}s, \tag{16}$$

$$\rho_t^\lambda = \int_t^\infty \varphi_s^\lambda\, \mathrm{d}s. \tag{17}$$

We shall first show how Theorem 2 follows from Lemma 4.

Proof of Theorem 2. By applying the Laplace transform to both sides of the equations (15)–(17) we get:

$$L_\theta(p) = \frac{\lambda + L_\varphi(p)}{2\lambda + p},$$

$$L_{\varphi}(p) = \frac{\lambda}{2}L_{\theta}(p) + \frac{1}{2}L_{\varphi}(p)L_{\theta}(p),$$
$$L_{\rho}(p) = \frac{\lambda - L_{\varphi}(p)}{p}.$$

The only solution of this system of equations which satisfies the natural conditions $\lim_{p\to\infty} L_X(p) = 0$ is

$$L_{\theta}(p) = 1 - p^{1/2}(p + 2\lambda)^{-1/2},$$
$$L_{\varphi}(p) = \lambda + p - p^{1/2}(p + 2\lambda)^{1/2},$$
$$L_{\rho}(p) = p^{-1/2}(p + 2\lambda)^{1/2} - 1.$$

Now, we can use the fact that the Laplace transform of a sum of two modified Bessel functions I_0 and I_1 is given by

$$L_{I_0(\lambda t)+I_1(\lambda t)}(p) = \frac{1}{\lambda}((p + \lambda)^{1/2}\,(p - \lambda)^{-1/2} - 1),$$

to invert the Laplace transforms of θ_t^{λ} and ρ_t^{λ} and obtain (6) and (8). The density of collisions φ_t^{λ} can be computed by taking the derivative of the particle density ρ_t^{λ}. The asymptotics of these probability densities are then obtained by the application of a Tauberian theorem ((Feller, 1968b), Theorem 4, p. 423). The first equation in (8) can be checked by comparing the expression for ρ_t^{λ} with (1). This completes the proof of Theorem 2. □

Proof of Lemma 4. First, we note that the first two statements of Lemma 3 have straightforward continuous analogs.

For given x, t, $\triangle x$ and $\triangle t$ let us denote by $M[x,t,\triangle x,\triangle t]$ the interior of the space–time region surrounded by the parallelogram with vertices $\{(x,t), (x+\triangle x,t), (x+\triangle x+\triangle t, t+\triangle t), (x+\triangle t, t+\triangle t)\}$, together with its lower and left boundaries $[(x,t), (x+\triangle x,t)) \cup [(x,t), (x+\triangle t, t+\triangle t))$. The area of $M[x,t,\triangle x,\triangle t]$ is $|\triangle x|\cdot\triangle t$, and hence,

$$P_{\lambda}(\text{there is a collision in } M[x,t,\triangle x,\triangle t]) = \varphi_t^{\lambda}|\triangle x|\triangle t + |\triangle x|\cdot o(\triangle t).$$

In the case $t=0$ the lemma is trivial.

Let us now fix some $t>0$, and take $\triangle t>0$, which shall later be shrunk to zero.

In order to prove (15) we take a look at the Palm version of the process, $\mathring{\xi}_{\bullet}^{\lambda}$, and assume, without loss of generality, that the particle starting at 0 went initially to the right.

Consider the event

$$C = \{\text{the number of particles at time 0 in the interval } [2t, 2t+2\triangle t) \text{ is at most } 1\}.$$

Note that $P(C) = 1 - O((\triangle t)^2)$.

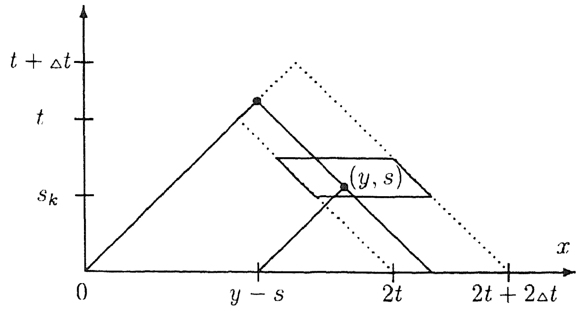

Fig. 1. Computation of θ_t^λ.

Fix $N \geqslant 1$, and break the interval $[0,t)$ into N subintervals of the length $\triangle s = t/N$ by the points $s_k = kt/N$, $0 \leqslant k \leqslant N$. Under the condition C we have (see Fig. 1)

$$\{\text{the particle starting at 0 has its first collision in } [t,\ t+\triangle t)\}$$

$$= (\{\text{the first particle to the right of 0 is at the time 0 in } [2t,\ 2t+2\triangle t)\}$$

$$\cap\, \{\text{it starts going to the left}\})$$

$$\cup \left(\bigcup_{k=0}^{N-1} (\{\text{there is a collision at some point } (s,y) \text{ in}\right.$$

$$M[2t+2\triangle t - s_k, s_k, -2\triangle t, \triangle s]\}$$

$$\cap\, \{\text{after this collision the particle goes to the left}\}$$

$$\left.\cap\, \{\text{at time 0 there are no particles between 0 and } y-s\})\right).$$

Now, we let first $\triangle s$ and then $\triangle t$ tend to 0. In the limit the complement of the condition C becomes negligible, and we obtain the integral equation (15).

To prove (16) we turn to the process $\xi_\bullet^\lambda$. For given x and t, we introduce the same partition of $[0,t)$ as above. Let $\triangle x$ be of order $\triangle t$. Now we shall condition on the event

$$D = \{\text{the number of particles at time 0 in the interval } [0,\triangle x) \text{ is at most 1}\}.$$

Given D, we have

$$\{\text{there is a collision in } M[x,t,\triangle x,\triangle t]\}$$

$$= (\{\text{there is a particle in } [0,\triangle x) \text{ at time 0}\}$$

$$\cap\, \{\text{it starts going to the right}\}$$

$$\cap \{\text{its first collision after time 0 takes place within the interval } [t,\ t+\triangle t)\})$$

$$\cup \left(\bigcup_{k=0}^{N-1} (\{\text{there is a collision at some time } s \text{ in } M[s_k, s_k, \triangle x, \triangle s]\}\right.$$

$$\cap \{\text{after this collision the particle goes to the right}\}$$

$$\cap \{\text{the interval between this and the next collision is in } [t-s, t-s+\triangle t)\}) \Big).$$

Now, we obtain the integral equation (16) by the same limiting procedure as above. □

Final remarks

- Our technique cannot be applied to higher dimensions. It seems unlikely that exact results can be obtained there for finite time, but perhaps asymptotics can be computed by a different technique. We can guess that in higher dimensions the particle density has asymptotic order of t^{-1}, with a possible logarithmic correction coefficient, as in the CRW case (Bramson and Griffeath, 1980a).
- More realistic models should have more than just two possible speeds with which a particle can move.

Acknowledgements

I would like to thank J. van den Berg and B. Tóth for their interest in this work, for stimulating discussions and useful remarks.

References

Arratia, R., 1981. Limiting point processes for rescaling of coalescing and annihilating random walks on $\mathbb{Z}^d$. Ann. Probab. 9, 909–936.

Belitsky, V., Ferrari, A., 1995. Ballistic annihilation and deterministic surface growth. J. Statist. Phys. 80, 517–543.

Bramson, M., Griffeath, D., 1980a. Asymptotics for interacting particle systems on $\mathbb{Z}^d$. Z. Wahrsch. verw. Geb. 53, 183–196.

Bramson, M., Griffeath, D., 1980b. Clustering and dispersion rates for some interacting particle systems on $\mathbb{Z}^1$. Ann. Probab. 8, 183–213.

Feller, W., 1968a. An Introduction to Probability Theory and its Applications, vol. I. Wiley, New York.

Feller, W., 1968b. An Introduction to Probability Theory and its Applications, vol. II. Wiley, New York.

Fisch, R., 1992. Clustering in the one-dimensional three-color cyclic cellular automaton. Ann. Probab. 20, 1528–1548.

Griffeath, D., 1979. Additive and Cancellative Interacting Particle Systems. Lecture Notes in Mathematics, vol. 724. Springer, New York.

Harris, T.E., 1976. On a class of set-valued Markov processes. Ann. Probab. 4, 175–194.

Liggett, T.M., 1985. Interacting Particle Systems. Springer, Berlin.